Information Retrieval

WS 2016 / 2017

Lecture 4, Tuesday November 15th, 2016
(Compression, Codes, Entropy)

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Overview of this lecture

- Organizational
  - Your experiences with ES3 Efficient List Intersection
- Compression
  - Motivation saves space **and** query time
  - Codes Elias, Golomb, Variable-Byte
  - Entropy Shannon's famous theorem
  - **Exercise Sheet 4:** three nice proofs → part of Shannon's theorem + optimality of Golomb + size of inverted index

    We take a break from implementation work this week

Experiences with ES3 1/2

- Summary / excerpts
  - Interesting exercise, many liked performance tweaking
  - Less work than ES2 again
  - Lack of programming practice in Java or C++
  - People who started late took much longer
  - Some found it hard to make an improvement
  - Given code already used native arrays and "while" trick
  - Some of you had large variation between runs
  - Coding while watching the US election results is even worse than lack of sleep, etc.

Experiences with ES3 2/2

- Results
  - Three inverted lists of different lengths

them	1,717,305 postings
existence	162,511 postings
bielefeld	5,257 postings

  - Query them+bielefeld, list length ratio = 327

    Any of galloping, skip ptrs, bin. search give large speedup

  - Query existence+bielefeld, list length ratio = 31

    Skipping helps, but not too much

  - Query them+existence, list length ratio = 11

    Skipping costs more than it helps, switch to tuned baseline

Compression 1/6

- Motivation
  - Inverted lists can become very large

    Recall: length of an inverted list of a word = total number of occurrences of that word in the collection

    For example, in the English Wikipedia:

them:	1,717,305 occurrences
year:	2,052,964 occurrences
one:	4,022,417 occurrences

  - Compression potentially saves space **and** time

Compression 2/6

- Index in **memory**
  - Then compression saves memory (obviously)
  - Also: the index might be too large to fit into memory without compression, and with compression it does

    Fitting in memory is good because reading from memory is much much **much** faster than reading from disk

    Transfer rate from memory ≈ 2 GB / second

    Transfer rate from disk ≈ 50 MB / second

Compression 3/6

- Index on **disk**:
  - Then compression saves disk space (obviously)
  - But it also saves query time, here is a realistic example:

Disk transfer time:	50 MB / second	
Compression rate:	Factor 5	
Decompression time:	30 MB / second	
Inverted list of size:	50 MB	
Reading uncompressed:	1.0 seconds	→ 50 MB
Reading compressed:	0.2 seconds	→ 10 MB
Decompressing:	0.3 seconds	→ 50 MB

Reading compressed + decompression **twice faster** compared to reading uncompressed

Compression 4/6

for web-scale collections
($>$ 4.2 billion $= 2^{32}$)
even 8 bytes per id

- Gap encoding
  - Example inverted list (doc ids only):

    3, 17, 21, 24, 34, 38, 45, ..., 11876, 11899, 11913, ...
  - Numbers small in the beginning, large in the end, using an int for each id would be **4 bytes per id**
  - Alternative: store differences from one item to next:

    +3, +14, +4, +3, +10, +4, +7, ..., +12, +23, +14, ...
  - This is called **gap encoding**
  - Works as long as we process the lists from left to right
  - Now we have a sequence of mostly (but not always) small numbers ... how do we store these in little space?

Compression 5/6

- Binary representation
  - We can write number x in binary using $\lfloor \log_2 x \rfloor + 1$ bits

x	binary	number of bits	
1	1	1	$\lfloor \log_2 1 \rfloor + 1 = 1$
2	10	2	$\lfloor \log_2 2 \rfloor + 1 = 2$
3	11	2	$\lfloor \log_2 3 \rfloor + 1 = 2$
4	100	3	$\lfloor \log_2 4 \rfloor + 1 = 3$
5	101	3	...

  - This encoding is optimal in a sense … see later slides
  - So why not just (gap-)encode like this and concatenate:

+3, +14, +4, … → 11, 1110, 100, … → 111110100…

Compression 6/6

- Prefix-free codes, definition
  - Decode bit sequence from the last slide: 111110100

This could be:	+3, +14, +4	→	11, 1110, 100
Could also be:	+7, +6, +4	→	111, 110, 100
Or:	+3, +3, +2, + 4	→	11, 11, 10, 100

  - Problem: we have no way to tell where one code ends and the next code begins

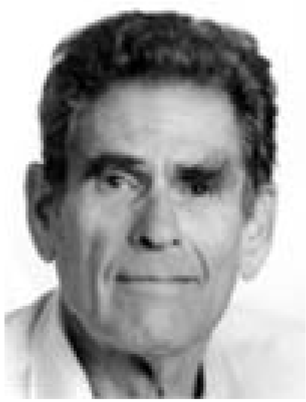
    Equivalently: some codes are prefixes of other codes
  - In a **prefix-free code**, no code is a prefix of another

    Then decoding from left to right is unambiguous !

Codes 1/4

- Elias-Gamma ... from 1975
  - Write $\lfloor \log_2 x \rfloor$ zeros, then x in binary like on slide 9
  - Prefix-free, because the number of initial zeros tells us exactly how many bits of the code come afterwards
  - Code for x has a length of exactly $2 \cdot \lfloor \log_2 x \rfloor + 1$ bits

1	1
2	010
3	011
4	00100
⋮	
10	0001010



Peter Elias
1923 – 2001

Codes 2/4

+ 1 because $\lfloor \log_2 1 \rfloor = 0$ and there is no code for 0 in Elias Gamma

- Elias-Delta ... also from 1975
  - Write $\lfloor \log_2 x \rfloor + 1$ in Elias-Gamma, followed by x in binary (like on slide 9) but **without** the leading 1
  - Elias-Delta is also prefix-free and the length of the code length is $\lfloor \log_2 x \rfloor + 2 \log_2 \log_2 x + O(1)$ bits

1	1
2	010 0
3	010 1
4	011 00
⋮	
10	00100 010

$\underbrace{00100}_{\lfloor \log_2 10 \rfloor + 1 \text{ in Elias-Gamma}}\ \underbrace{010}_{10 \text{ in binary (1010) without the leading 1}}$

Elias-Gamma Codes

1	1
2	010
3	011
4	00100
...	
10	0001010

Codes 3/4

q = quotient

r = remainder

- Golomb (not Gollum) ... from 1966
  - Comes with an integer parameter M, called **modulus**
  - Write x as q · M + r, where q = x div M and r = x mod M
  - The code for x is then the concatenation of:
    - q written in unary with 0s — q bits
    - a single 1 (as a delimiter) — 1 bit
    - r written in binary — $\lceil \log_2 M \rceil$ bits

Solomon Golomb
1932 – 2016

$M = 16$, $x = 67 = \underbrace{4}_{=:q} \cdot 16 + \underbrace{3}_{=:r}$

$\underbrace{0000}_{q \text{ in unary}}\,1\,\underbrace{0011}_{\text{3 in binary with fixed width of } \lceil \log_2 M \rceil \text{ bits}}$

Codes 4/4

- Variable-Byte (VB)
  - Idea: use **whole bytes**, in order to avoid the (expensive) bit fiddling needed for the previous schemes

    VB often used in practice, for exactly that reason
  - Use one bit of each byte to indicate whether this is the last byte in the current code or not
  - VB is also used for UTF-8 encoding … see later lecture

$x = 501 = 3 \cdot 128 + 117$

write a 2 bytes $1\underbrace{0000011}_{\text{code word 1}}$ $0\underbrace{1110101}_{\text{code word 2}}$

continuation bit

1 means: more bytes will follow

0 means: last byte of code

0000011|1110101

should be 501

Entropy 1/12

- Motivation
  - Which code compresses the best ?

    It depends !

    But on what ?
  - Roughly: it depends, on the relative frequency on the numbers / symbols we want to encode

    For example, in natural language, an "e" is much more frequent than a "z"

    So we should encode "e" with less bits than "z"
  - The next slides will make this more precise

Entropy 2/12

we take $0 \cdot \log_2 0 := 0$

- Entropy
  - **Intuitively:** the information content of a message = the optimal number of bits to encode that message
  - **Formally:** defined for a discrete random variable X

    Without loss of generality range of $X = \{1, ..., m\}$

    Think of X as generating the symbols of the message

    Then the **entropy** of X is written and defined as

    $H(X) = - \Sigma_i \, p_i \log_2 p_i$ where $p_i = Prob(X = i)$
  - Example 1: one $p_i = 1$ all other 0, then $H(X) = 0$
  - Example 2: all $p_i = 1/m$, then $H(X) = \log_2 m$

    $= -\sum_{i=1}^{m} \frac{1}{m} \cdot \log_2 \frac{1}{m} = m \cdot \frac{1}{m} \cdot \log_2 m$

Entropy 3/12

- Shannon's source coding theorem ... from 1948
  - Let X be a random variable with finite range
  - For an arbitrary prefix-free (**PF**) encoding, let L_x be the length of the code for $x \in \text{range}(X)$

    (1) For any PF encoding it holds: $\mathbf{E}\, L_X \geq H(X)$

    (2) There is a PF encoding with: $\mathbf{E}\, L_X \leq H(X) + 1$

    where **E** denotes the expectation

    In words: no code can be better than the entropy, and there is always a code as good

almost
(because of
the + 1)

Claude Shannon
1916 – 2001

Entropy 4/12

- Central Lemma ... to prove the source coding theorem
  - Denote by L_i the length of the code for the i-th symbol, then

    (1) Given a PF code with lengths L_i ⇨ $\Sigma_i\, 2^{-L_i} \leq 1$

    (2) Given L_i with $\Sigma_i\, 2^{-L_i} \leq 1$ ⇨ exists PF code with length L_i

  - Note: $\Sigma_i\, 2^{-L_i} \leq 1$ is known as "Kraft's inequality"
  - Intuitively: not all L_i can be small ... small L_i → large 2^{-L_i}

    For example, the lemma says that a prefix-free code where three $L_i = 1$ is not possible, because $2^{-1} + 2^{-1} + 2^{-1} > 1$

Entropy 5/12

Lets say the encoding scheme is Elias Gamma

Random Experiment: 0 1 1 $Pr = 2^{-3}$

$Pr = 2^{-5}$ 0 0 1 0 1

- Proof of central lemma, part (1)
  - Show: given PF code with lengths L_i then $\Sigma_i\, 2^{-L_i} \le 1$
  - Consider the following random experiment:

    Generate a random binary sequence, and pick each bit independent from all other bits

    Stop when you have a valid code, or when no more code is possible … well-defined for PF codes only !
  - Let C_i = the event that code i is generated → $Pr(C_i) = 2^{-L_i}$
  - Then $\underbrace{Pr(C_1)}_{=2^{-L_1}} + \ldots + \underbrace{Pr(C_m)}_{2^{-L_m}} = Pr(C_1 \cup \ldots \cup C_m) \le 1$
  - And the left-hand side is just $\Sigma_i\, 2^{-L_i}$

Entropy 6/12

$L_1 = 1$, $L_2 = 2$, $L_3 = 3$, $L_4 = 3$

Then $\sum_{i=1}^{4} 2^{-L_i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1 \le 1$

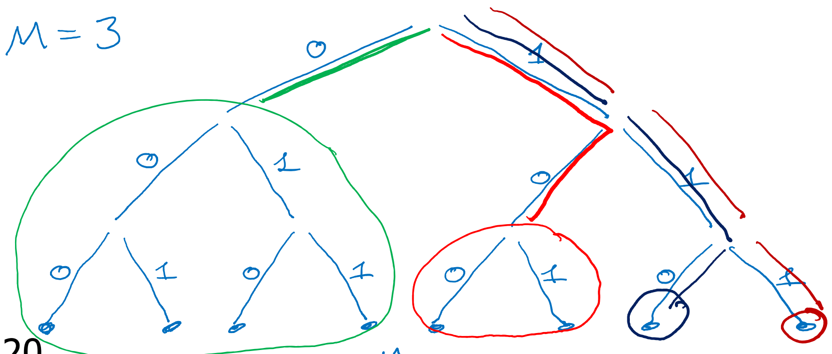
- Proof of central lemma, part (2)

  This is called Huffman encoding

  - To show: L_i with $\Sigma_i\ 2^{-L_i} \le 1$ ⇨ exists PF code with lengths L_i
  - Complete binary tree of depth $M = \max L_i$ … has 2^M leaves

    in the example $M = 3$
  - Mark all left edges 0, and all right edges 1
  - Consider the code lengths L_i in sorted order, smallest first
  - Then iterate: pick subtree with $2^{M - L_i}$ leaves that does not overlap with already picked subtrees … path to that subtree gives code for symbol i and sum $2^{M - L_i} = 2^M \cdot \Sigma_i\ 2^{-L_i} \le 2^M$

$M = 3$



$2^M = 8$ leaves

$L_1 = 1$, $2^{M-L_i} = 4$, code = 0

$L_2 = 2$, $2^{M-L_2} = 2$, code = 10

$L_3 = 3$, $2^{M-L_3} = 1$, code = 110

$L_4 = 3$, $2^{M-L_4} = 1$, code = 111

Entropy 7/12

- Proof of source coding theorem, part (1)
  - To show: for any PF encoding $\mathbf{E}\, L_X \geq H(X)$
  - By definition of expectation: $\mathbf{E}\, L_X = \Sigma_i\, p_i \cdot L_i$ (1)
  - By Kraft's inequality: $\Sigma_i\, 2^{-L_i} \leq 1$ (2)
  - Using Lagrange, it can be shown that, under the constraint (2), (1) is **min**imized for $L_i = \log_2 1/p_i$
  - Then $E\, L_X = \Sigma_i\, p_i \cdot L_i \geq \Sigma_i\, p_i \cdot \log_2 1/p_i = H(X)$

    This is Exercise 1 from ES4

    Perfect exercise to practice Lagrangian optimization and deepen understanding of the source coding theorem

Entropy 8/12

$$\sum_{i=1}^{m} 2^{-L_i} = \sum_{i=1}^{m} 2^{-\lceil \log_2 \frac{1}{p_i} \rceil} \le \sum_{i=1}^{m} 2^{-\log_2 \frac{1}{p_i}} = \sum_{i=1}^{m} p_i = 1$$

$$-\log_2 \frac{1}{p_i} = \log_2 p_i$$

- Proof of source coding theorem, part (2)
  - Show: there is a PF encoding with $\mathbf{E}\, L_X \le H(X) + 1$
  - Let $L_i = \lceil \log_2 1/p_i \rceil$, then $\Sigma_i\, 2^{-L_i} \le 1$

    Note that rounding is necessary because the code length must be an integer, and that we need to round upwards, so that Kraft's inequality holds
  - By the central lemma, part (2), there then exists a PF code with code lengths L_i
  - By definition of expectation: $\mathbf{E}\, L_X = \Sigma_i\, p_i \cdot L_i$
  - Hence $\mathbf{E}\, L_X = \Sigma_i\, p_i \cdot \lceil \log_2 1/p_i \rceil \le \Sigma_i\, p_i \cdot (\log_2 1/p_i + 1)$

    $= \Sigma_i\, p_i \cdot \log_2 1/p_i + \Sigma_i\, p_i = H(X) + 1$

Entropy 9/12

- Entropy-optimal codes
  - Consider a PF code with L_i = code length for symbol i and p_i = probability for symbol i
  - We say that the code is optimal for distribution p_i if

    $L_i \leq \log_2 1/p_i + 1$

    Then $\mathbf{E}\, L_X \leq H(X) + 1$ and by Shannon's theorem this is the best we can hope for

    For the optimality proof from Exercise 2 from ES4, it suffices that you show $L_i \leq \log_2 1/p_i + \mathbf{O(1)}$

Entropy 10/12

- Universal codes
  - A prefix-free code is called **universal** if for <u>every</u> probability distribution over the symbols to be encoded

    $\mathbf{E}\ L_X = O(\ H(X)\ )$

    That is, the expected code length is within a constant factor of the optimum for <u>any</u> distribution
  - Elias-Gamma, Elias-Delta, Golomb, and Variable-Byte are all universal in this sense

    For a finer distinction, the definition of optimality from the previous slide is better

    $\mathbf{E}\ L_X \leq H(X) + 1$ versus $\mathbf{E}\ L_X = O(\ H(X)\ )$

Entropy 11/12

- Entropy-optimality of Elias-Gamma
  - Recall: code length for Elias-Gamma is $L_i = 2 \lfloor \log_2 i \rfloor + 1$
  - For which probability distribution is this entropy-optimal?
  - We need $L_i = 2 \lfloor \log_2 i \rfloor + 1 \leq \log_2 1/p_i + 1$
  - This suggests something like $p_i \approx 1 / i^2$ because:

    $p_i = 1 / i^2 \;\rightarrow\; \log_2 1/p_i = \log_2 i^2 = 2 \cdot \log_2 i$
  - We have to take care that the p_i sum to 1, hence let $p_i = 1 / i^2$ for $i \geq 2$, and p_1 such that $\Sigma_i \, p_i = 1$

    That is, numbers $i \geq 2$ occur with probability $1 / i^2$

    Note that $\Sigma_{i=1..\infty} 1 / i^2 = \pi^2 / 6 = 1.6449\ldots$

Entropy 12/12

- Optimality of Golomb
  - Consider the following random experiment for the generation of an inverted list L of length m :

    Include each document in L with probability $p = m/n$, independently of each other, where n = #documents
  - Let X be a fixed **gap** in this inverted list, then

    $\Pr(X = x) = (1 - p)^{x-1} \cdot p =: p_x$ for $x = 1, 2, 3, \ldots$

    Exercise 2 from ES4: Golomb is optimal for this distrib.
  - Bottom line: Golomb is optimal for gap-encoded lists

    But not practical, because of the bit fiddling, see slide 14

References

- Textbook

  Section 5: Index compression

  Section 5.3: Postings file compression   some codes only

- Wikipedia

  http://en.wikipedia.org/wiki/Elias_gamma_coding

  http://en.wikipedia.org/wiki/Elias_delta_coding

  http://en.wikipedia.org/wiki/Golomb_coding

  http://en.wikipedia.org/wiki/Variable-width_encoding

  http://en.wikipedia.org/wiki/Source_coding_theorem

  http://en.wikipedia.org/wiki/Kraft_inequality