Information Retrieval

WS 2015 / 2016

Lecture 4, Tuesday November 10th, 2015
(Compression, Codes, Entropy)

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Overview of this lecture

- Organizational
  - Your experiences with ES3 — Efficient List Intersection
  - Assistant: Björn → Elmar
- Compression
  - Motivation — saves space **and** query time
  - Codes — Elias, Golomb, Variable-Byte
  - Entropy — Shannon's theorem
  - **Exercise Sheet 4:** prove the optimality of some of the codes + investigate the limits of what can be achieved

    We take a break from implementation work this week

Experiences with ES3 1/2

- Summary / excerpts
  - Interesting exercise, many liked performance tweaking
  - Submissions are getting less … not unusual though
  - Sentinels already implemented in C++, but not in Java
  - Not so easy to improve on a tuned baseline
  - Complex improvements can cost more than they help

    E.g. galloping always slower than binary search for some
  - Skip pointers can be simulated via a simple offset

    Like a low-budget version of galloping search
  - Seemingly insignificant code changes can affect runtime

Experiences with ES3 2/2

- Results
  - Three inverted lists of different lengths

film	171,951 postings	repeated 100 times
comedy	27,706 postings	repeated 100 times
2015	285 postings	repeated 100 times

  - Query film+2015, list length ratio = 603

    Any of galloping, skip ptrs, bin. search give large speedup
  - Query comedy+2015, list length ratio = 97

    Skipping helps, but not too much
  - Query film+comedy, list length ratio = 6

    Skipping costs more than it helps, switch to tuned baseline

Compression 1/6

- Motivation
  - Inverted lists can become very large

    Recall: length of an inverted list of a word = total number of occurrences of that word in the collection

    For example, in the English Wikipedia:

    film: 1,667,147 occurrences
    year: 2,052,964 occurrences
    one: 4,022,417 occurrences

  - Compression potentially saves space **and** time

Compression 2/6

- Index in **memory**
  - Then compression saves memory (obviously)
  - Also: the index might be too large to fit into memory without compression, and with compression it does

    Fitting in memory is good because reading from memory is much much **much** faster than reading from disk

    Transfer rate from memory ≈ 2 GB / second

    Transfer rate from disk ≈ 50 MB / second

Compression 3/6

when combined with zipping, can skip compressed parts with- and even decompressing them

- Index on **disk**:
  - Then compression saves disk space (obviously)
  - But it also saves query time, here is a realistic example:

Disk transfer time:	50 MB / second	
Compression rate:	Factor 5	
Decompression time:	30 MB / second	
Inverted list of size:	50 MB	
Reading uncompressed:	1.0 seconds	→ 50 MB
Reading compressed:	0.2 seconds	→ 10 MB
Decompressing:	0.3 seconds	→ 50 MB

Reading compressed + decompression **twice faster** compared to reading uncompressed

Compression 4/6

for web-scale even 8 bytes / id

- Gap encoding
  - Example inverted list (doc ids only):

    3, 17, 21, 24, 34, 38, 45, ..., 11876, 11899, 11913, ...

  - Numbers small in the beginning, large in the end, using an int for each id would be **4 bytes per id**
  - Alternative: store differences from one item to next:

    +3, +14, +4, +3, +10, +4, +7, ..., +12, +23, +14, ...

  - This is called **gap encoding**
  - Works as long as we process the lists from left to right
  - Now we have a sequence of mostly (but not always) small numbers ... how do we store these in little space?

Compression 5/6

- Binary representation
  - We can write number x in binary using $\lfloor \log_2 x \rfloor + 1$ bits

x	binary	number of bits	
1	1	1	$= \lfloor \overset{=0}{\log_2 1} \rfloor + 1$
2	10	2	$= \lfloor \overset{=1}{\log_2 2} \rfloor + 1$
3	11	2	$= \lfloor \overset{=1}{\log_2 3} \rfloor + 1$
4	100	3	$= \lfloor \overset{=2}{\log_2 4} \rfloor + 1$
5	101	3	

  - This encoding is optimal in a sense … see later slides
  - So why not just (gap-)encode like this and concatenate:

    +3, +14, +4, … → 11, 1110, 100, … → 111110100…

Compression 6/6

- Prefix-free codes, definition
  - Decode bit sequence from the last slide: 111110100

    This could be: +3, +14, +4 → 11, 1110, 100

    Could also be: +7, +6, +4 → 111, 110, 100

    Or: +3, +3, +2, + 4 → 11, 11, 10, 100
  - Problem: we have no way to tell where one code ends and the next code begins

    Equivalently: some codes are prefixes of other codes
  - In a **prefix-free code**, no code is a prefix of another

    Then decoding from left to right is unambiguous

Codes 1/4

needs $\lfloor \log_2 x \rfloor + 1$ bits

- Elias-Gamma ... from 1975
  - Write $\lfloor \log_2 x \rfloor$ zeros, then x in binary like on slide 9
  - Prefix-free, because the number of initial zeros tells us exactly how many bits of the code come afterwards
  - Code for x has a length of exactly $2 \cdot \lfloor \log_2 x \rfloor + 1$ bits

$1 \rightarrow 1$

$2 \rightarrow 010$

$3 \rightarrow 011$

$4 \rightarrow 00100$

$\vdots$

$10 \rightarrow 0001010$

Peter Elias
1923 – 2001

Codes 2/4

+1 because there is no EG code for 0

- Elias-Delta ... also from 1975
  - Write $\lfloor \log_2 x \rfloor + 1$ in Elias-Gamma, followed by x in binary (like on slide 9) but **without** the leading 1
  - Elias-Delta is also prefix-free and the length of the code length is $\lfloor \log_2 x \rfloor + 2 \log_2 \log_2 x + O(1)$ bits

Pr. in Exercise 4.1

$\lfloor \underbrace{\log_2 4}_{=2} \rfloor + 1 = 3$ in EG

Elias-Gamma

1 → 1
2 → 010
3 → 011
4 → 00100

1 → 1
2 → 0100 0
3 → 0101 1
4 → $\boxed{011}$ $\boxed{00}$ — 4 in binary w/o leading 1
⋮
10 → 00100 010

Codes 3/4

q = "quotient"
r = "remainder"

prefix free

ALSO PF
think about it

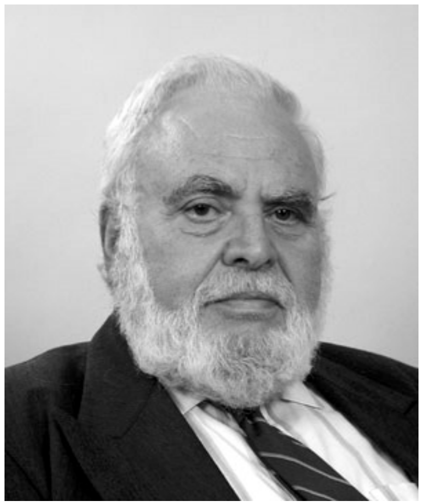
- Golomb (not Gollum) ... from 1966
  - Comes with an integer parameter M, called **modulus**
  - Write x as $q \cdot M + r$, where $q = x \text{ div } M$ and $r = x \bmod M$
  - The code for x is then the concatenation of:
    - q written in unary with 0s — q bits
    - a single 1 (as a delimiter) — 1 bit
    - r written in binary — $\lceil \log_2 M \rceil$ bits

$M = 16$, $x = 70 = \underset{q}{4} \cdot 16 + \underset{r}{6}$

Code for x is 0000 1 $\underbrace{0110}$

FIXED LENGTH for given M

Solomon Golomb
1932 – still alive



Codes 4/4

7 bits of „information" per byte

- Variable-Byte (VB)
  - Idea: use **whole bytes**, in order to avoid the (expensive) bit fiddling needed for the previous schemes

    VB often used in practice, for exactly that reason
  - Use one bit of each byte to indicate whether this is the last byte in the current code or not
  - VB is also used for UTF-8 encoding … see later lecture

$x = 521 = 4 \cdot 128 + 9$

Byte 1 Byte 2

VB-code for x $\boxed{1|0000100}$ $\boxed{0|0001001}$

0000100 : 0001001

would be 521 in bin.

↑ indicates that code continues

4 in binary

↑ indicates that last byte of code

Entropy 1/12

- Motivation
  - Which code compresses the best ?

    It depends !

    But on what ?
  - Roughly: it depends, on the relative frequency on the numbers / symbols we want to encode

    For example, in natural language, an "e" is much more frequent than a "z"

    So we should encode "e" with less bits than "z"
  - The next slides will make this more precise

Entropy 2/12

$H(X) \geq 0$

$H(X) = 0$ iff one $p_i = 1$ all other zero

Note: for H, we define $0 \cdot \log_2 0 = 0$

- Entropy
  - **Intuitively:** the information content of a message = the optimal number of bits to encode that message
  - **Formally:** defined for a discrete random variable X

    Without loss of generality range of X = {1, ..., m}

    Think of X as generating the symbols of the message

    Then the **entropy** of X is written and defined as

    $H(X) = - \Sigma_i \, p_i \log_2 p_i$ where $p_i = Prob(X = i)$

    E.g. m symbols, each equally likely $\Rightarrow p_i = \frac{1}{m}$

    $\Rightarrow H(X) = -\sum_{i=1}^{m} \frac{1}{m} \cdot \log_2 \frac{1}{m} = \log_2 m$

Entropy 3/12

- Shannon's source coding theorem ... from 1948
  - Let X be a random variable with finite range
  - For an arbitrary prefix-free (**PF**) encoding, let $L(x)$ be the length of the code for $x \in \text{range}(X)$

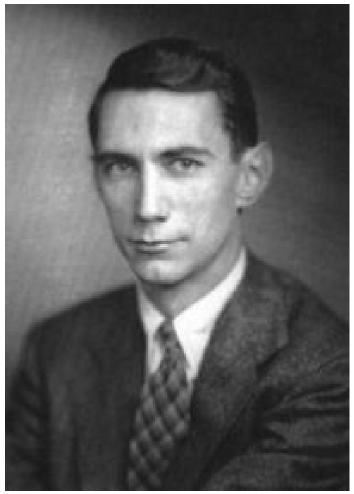
    (1) For any PF encoding it holds: $\mathbf{E}\, L(X) \geq H(X)$

    (2) There is a PF encoding with: $\mathbf{E}\, L(X) \leq H(X) + 1$

    where **E** denotes the expectation

    In words: no code can be better than the entropy, and there is always a code as good

    almost



Claude Shannon
1916 – 2001

Entropy 4/12

Intuitively: not all L_i can be small (small $L_i \rightarrow$ large 2^{-L_i})

E.g. $\underset{2^{-1}}{L_1 = 1}$, $\underset{2^{-1}}{L_2 = 1}$, $\underset{2^{-1}}{L_3 = 1}$, ... NOT POSSIBLE

- **Central Lemma** ... to prove the source coding theorem
  - Denote by L_i the length of the code for the i-th symbol, then

    (1) Given a PF code with lengths L_i ⇨ $\Sigma_i 2^{-L_i} \leq 1$

    (2) Given L_i with $\Sigma_i 2^{-L_i} \leq 1$ ⇨ exists PF code with length L_i

  - Note: $\Sigma_i 2^{-L_i} \leq 1$ is known as "Kraft's inequality"

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Entropy 5/12

01101 $L_i = 5$

$\underbrace{(\frac{1}{2}) \cdot \ldots (\frac{1}{2})}_{5 \text{ terms}} = 2^{-L_i}$

- Proof of central lemma, part (1)
  - To show: given a PF code with lengths L_i ⇨ $\Sigma_i\, 2^{-L_i} \leq 1$
  - Consider the following random experiment:

    Generate a random binary sequence, and pick each bit independent from all other bits 0110 STOP

    Stop when you have a valid code, or when no more code is possible … well-defined for PF codes only !

    code for symbol i
  - Let C_i be the event that code i is generated

$\Pr(C_1 \dot\cup C_2 \dot\cup \ldots \dot\cup C_m) \leq 1$

$= \Pr(C_1) + \Pr(C_2) + \ldots + \Pr(C_m) = \sum_{i=1}^{m} 2^{-L_i}$

$\Pr(C_i) = 2^{-L_i}$

Entropy 6/12

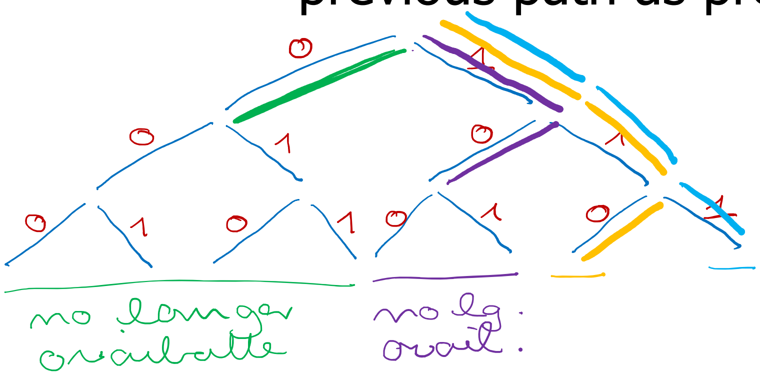
GIVEN:
$L_1 = 1,\ L_2 = 2,\ L_3 = 3,\ L_4 = 3$
$\sum 2^{-L_i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$ o.k.

- Proof of central lemma, part (2)

  ensures that there is always enough of the tree "left"

  - To show: L_i with $\Sigma_i\ 2^{-L_i} \leq 1$ ⇨ exists PF code with length L_i
  - Consider a complete binary tree of depth $\max L_i$ $= 3$
  - Mark all left edges 0, and all right edges 1
  - Consider the code lengths L_i in sorted order, smallest first
  - Then iterate: pick a path of length L_i from the root, with no previous path as prefix … this gives a PF code for symbol i

This principle is called Huffman encoding



$L_1 = 1,\ L_2 = 2,\ L_3 = 3,\ L_4 = 3$

Code for 1: 0
Code for 2: 10
Code for 3: 110
Code for 4: 111

Entropy 7/12

$p_1, \dots, p_m$ = prob. distr. over symbols

$\Sigma p_i = 1$

- Proof of source coding theorem, part (1)
  - To show: for any PF encoding **E** L(X) ≥ H(X)
  - By definition of expectation: **E** L(X) = $\Sigma_i\ p_i \cdot L_i$ (1)
  - By Kraft's inequality: $\Sigma_i\ 2^{-L_i} \le 1$ (2)
  - Using Lagrange, it can be shown that, under the constraint (2), (1) is **min**imized for $L_i = \log_2 1/p_i$

"average code length"

constraint

$$\Rightarrow \underline{E\,L(X)} = \Sigma_i\, p_i \cdot L_i \geq \Sigma_i\, p_i \cdot \log_2 1/p_i \underline{= H(X)}$$

LAGRANGE (sketch):

Exercise 4.4

$$\mathcal{L} = \sum_{i=1}^{m} p_i L_i + \lambda \left(1 - \sum_{i=1}^{m} 2^{-L_i}\right) \qquad 2^{-L_i} = e^{-\ln 2 \cdot L_i}$$

$$\frac{\partial \mathcal{L}}{\partial L_i} = p_i + \lambda \cdot \ln 2 \cdot 2^{-L_i} \stackrel{!}{=} 0 \quad \Rightarrow \dots \Rightarrow L_i = \log_2 \frac{1}{p_i}$$

Entropy 8/12

Given $p_1, \ldots, p_m$

- Proof of source coding theorem, part (2)
  - Show: there is a PF encoding with **E** $L(X) \leq H(X) + 1$
  - Let $L_i = \lceil \log_2 1/p_i \rceil$, then $\Sigma_i \, 2^{-L_i} \leq 1$

    $\log_2 1/p_i$ is the "ideal" code length

    Note that rounding is necessary because the code length must be an integer, and that we need to round upwards, so that Kraft's inequality holds
  - By the central lemma, part (2), there then exists a PF code with code lengths L_i
  - By definition of expectation: **E** $L(X) = \Sigma_i \, p_i \cdot L_i$

$$E\,L(X) = \sum_{i=1}^{m} p_i \cdot \lceil \log_2 \tfrac{1}{p_i} \rceil \leq \underbrace{\sum_{i=1}^{m} p_i \cdot \log_2 1/p_i}_{= H(X)} + \underbrace{\sum_{i=1}^{m} p_i}_{= 1}$$

$$\lceil \log_2 \tfrac{1}{p_i} \rceil \leq \log_2 \tfrac{1}{p_i} + 1$$

Entropy 9/12

- Entropy-optimal codes
  - Consider a PF code with L_i = code length for symbol i and p_i = probability for symbol i
  - We say that the code is optimal for distribution p_i if

    $L_i \leq \log_2 1/p_i + 1$

    Then **E** $L(X) \leq H(X) + 1$ and by Shannon's theorem this is the best we can hope for

    For the optimality proofs from Exercise Sheet 4, it suffices that you show $L_i \leq \log_2 1/p_i + \mathbf{O(1)}$

Entropy 10/12

- Universal codes
  - A prefix-free code is **universal** if for <u>every</u> probability distribution over the symbols to be encoded

    E L(X) = O(H(X))

    That is, the expected code length is within a constant factor of the optimum for <u>any</u> distribution

  - Elias-Gamma, Elias-Delta, Golomb, and Variable-Byte are all universal in this sense

    For a finer distinction, the definition of optimality from the previous slide is better

    E L(X) ≤ H(X) + 1 versus E L(X) = O(H(X))

Entropy 11/12

$p_i = 1/i^2$

$\Rightarrow \log_2 \frac{1}{p_i} = \log_2 i^2$

$= 2 \cdot \log_2 i$

- Optimality of Elias-Gamma
  - Recall: code length for Elias-Gamma is $L_i = 2 \lfloor \log_2 i \rfloor + 1$
  - For which probability distribution is this entropy-optimal?
  - We need $L_i = 2 \lfloor \log_2 i \rfloor + 1 \leq \log_2 1/p_i + 1$
  - This suggests something like $p_i \approx 1 / i^2$
  - Let $p_i = 1 / i^2$ for $i \geq 2$, and p_1 such that $\Sigma_i \, p_i = 1$

    That is, numbers $i \geq 2$ occur with probability $1 / i^2$

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} = 1.6\ldots$$

Entropy 12/12

- Optimality of Golomb
  - Consider the following random experiment for the generation of an inverted list L of length m :

    Include each document i in L with probability p = m/N, independently of each other, where N = #documents
  - Let G be a fixed **gap** in this inverted list, then

    $\Pr(G = i) = (1 - p)^{i-1} \cdot p =: p_i$ for $i = 1, 2, 3, \ldots$
  - Exercise 4.3: prove that Golomb is optimal for this distr.
  - Bottom line: Golomb is optimal for gap-encoded lists

    But not practical, because of the bit fiddling, see slide 14

References

- Textbook

  Section 5: Index compression

  Section 5.3: Postings file compression some codes only

- Wikipedia

  http://en.wikipedia.org/wiki/Elias_gamma_coding

  http://en.wikipedia.org/wiki/Elias_delta_coding

  http://en.wikipedia.org/wiki/Golomb_coding

  http://en.wikipedia.org/wiki/Variable-width_encoding

  http://en.wikipedia.org/wiki/Source_coding_theorem

  http://en.wikipedia.org/wiki/Kraft_inequality