$a_1 \geq a_2 \geq \dots \geq a_l > 0$ $a_i \in \mathbb{N}$

$$\sum_{j=1}^{l-1} \frac{a_j - a_{j+1}}{a_j} \leq 1 + \ln(a_1)$$

Obs 1: if $a_j = a_{j+1}$ then the respective term in the sum is 0

$\Rightarrow$ w.l.o.g.: $a_i \neq a_j$ if $i \neq j$

$$\sum_{j=1}^{l-1} \frac{a_j - a_{j+1}}{a_j} = \sum_{j=1}^{l-1} \left(1 - \frac{a_{j+1}}{a_j}\right)$$
$$\leq \sum_{j=1}^{l-1} \left(1 - \frac{a_{j+1}}{a_1}\right)$$

$$\Rightarrow \sum_{i=1}^{a_1} \frac{1}{i} = H(a_1) \leq \ln(a_1) + 1$$

$$\ln(n) \leq H(n) \leq \ln(n) + 1$$

() sum over every term between $\frac{1}{a_1}, \dots, 1$

t.s. $\displaystyle\sum_{j=1}^{l-1} \left(1 - \frac{a_{j+1}}{a_1}\right) \leq \sum_{i=1}^{a_1} \frac{1}{i}$

() all terms in the first sum are distinct from each (Obs 1) and all are in the interval

$\frac{1}{a_1}, \ldots, 1$

max: $a_j + 1 = a_1 - 1$

$$1 - \frac{a_j+1}{a_1} = 1 - \frac{a_1 - 1}{a_1} = 1 - 1 + \frac{1}{a_1}$$
$$= \frac{1}{a_1}$$

min: $a_j + 1 = 1$

$$1 - \frac{1}{a_1} \leq 1$$

Ski Rental Problem

rent skies for 1€ /day

Or buy them for M€

Online problem: decide on a daily basis whether to rent or to buy skies (depending e.g. on weather cond.)

What would be a good strategy?

t = # of skiing days a posteriori

$OPT = \min(t, M)$

• STRATEGY 1: buy immediately:

ratio (worst) $\frac{M}{0} = \infty$

$\left(\frac{M}{1} = M \right)$

if buying on a good day

• STRATEGY 2: only rent

ratio if $OPT = M$ $\frac{t}{M}$

for $t \gg M$ arbitrary bad

• STRATEGY 3: rent at most M days, buy then

cost of this strategy $\leq 2M$

optimal costs are M

ratio ≤ 2

(if $t < M$, then we are optimal anyway)

Theorem: Every det. Strategy for the Ski-Rental-Problem is at least 2-comp.

Prove: det. alg. "buy after T days"
(If $T = \infty$, arbitrary worse)

take a series σ of length T

$$\mathrm{cost}(\sigma) = T + M$$
$$\mathrm{OPT}(\sigma) = \min(T, M)$$

$$\frac{\mathrm{cost}(\sigma)}{\mathrm{OPT}(\sigma)} = \frac{T+M}{\min(T,M)} = \frac{T}{\min(T,M)} + \frac{M}{\min(T,M)}$$
$$\geq 1 + 1$$
$$= 2$$

<u>Randomized Ski-Rental</u>

STRATEGY: throw a coin

if 'HEAD' buy after $\frac{M}{2}$ days

if 'TAIL' buy after M days

Analysis: worst-case: good weather stops as soon as we have our own skies

↳ Case A: σ of length M

$$OPT(\sigma) = M$$

(C - cost of the strategy)

$$E(C(\sigma)) = \frac{1}{2} \cdot 2M + \frac{1}{2} \cdot \frac{3}{2} M$$

$$= \frac{7}{4} M \qquad \text{ratio } \frac{7}{4}$$

Case B: σ of length $\frac{M}{2}$

$$OPT(\sigma) = \frac{M}{2}$$

$$E(C(\sigma)) = \frac{1}{2} \cdot \frac{M}{2} + \frac{1}{2} \frac{3}{2} M$$

$$= M \qquad \text{ratio } 2$$

BETTER STRATEGY:

with prob. $\frac{1}{2}$ buy at time
$t = \alpha M$, $\alpha < 1$ and
with prob $\frac{1}{2}$ buy at M

σ of length M

$$OPT(\sigma) = M$$

$$E(C(\sigma)) = \frac{1}{2}(\alpha+1)M + \frac{1}{2}\cdot 2M = \frac{\alpha+3}{2}M$$

ratio $\frac{\alpha+3}{2}$

σ of length αM

$$OPT(\sigma) = \alpha M$$

$$E(C(\sigma)) = \frac{1}{2}(\alpha+1)M + \frac{1}{2}\alpha M = \frac{2\alpha+1}{2}M$$

ratio $1 + \frac{1}{2\alpha}$

What is the best α?

$$\min\left(\max\left(\frac{3+\alpha}{2},\ 1+\frac{1}{2\alpha}\right)\right)$$

$$\Leftrightarrow \frac{3+\alpha}{2} = 1+\frac{1}{2\alpha}$$

$$\Rightarrow \alpha^2+\alpha-1=0$$

$$\alpha = \frac{\sqrt{5}-1}{2}$$

$\Rightarrow$ comp. ratio: $\approx 1.809 < 2$

(Even better: decide on daily basis whether to buy by tossing a coin $\Rightarrow$ comp. ratio of 1.58 $= \frac{e}{e-1}$)

Bahn Card Problem

C, T, β

C – cost of buying a Bahn Card

T – interval in which the BC is valid

β – price reduction in [0,1]

BC 50 $\beta = \frac{1}{2}$

BC 25 $\beta = 0.75$

Question: At which point in time does it make sense to buy a BC if nothing is known about the travel behavior in the future?

BC(C, T, β)

(Ski-Rental problem: (M, ∞, 0))

$2 - \beta$

STRATEGY:

BC 50 • If costs exceed 272€, then buy

(136€, 1 year, 0.5) (proof: $C \to$ strat cost: $2C + \frac{1}{2}P$, opt: $C + \frac{1}{2}P$)

[• NEVER buying : comp. ratio $\frac{1}{\beta}$]

SAM alg.

buy tickets without BC until their summed price exceeds C, then buy the BC (before the last journey)

$\sigma = \sigma_1, \sigma_2$ travel requests

p_1, p_2 prices

t_1, t_2 departure times

BC bought at time t is valid in $[t, t+T)$

Cost of the strategy for σ:

$$\mathrm{Cost}(\sigma) = \begin{cases} \beta p_i & \text{if BC is valid at } t_i \\ p_i & \text{otherwise} \end{cases}$$

We call βp_i the reduced price of the ticket. Accordingly, σ_i is a reduced request if we have a valid BC at the time t_i (otherwise 'regular' request)

If we buy a BC at times

$\tau_1 < \tau_2 < \dots < \tau_n$ then we

call the sequence $T(\sigma)$ the
$= \tau_1, \dots, \tau_n$

B-schedule of the strategy

on σ. We say the length of

the B-schedule is $|T(\sigma)|$. The total

cost on σ is then

$$\mathrm{cost}(\sigma) = |T(\sigma)| \cdot C + \sum_i \mathrm{cost}\ \sigma(i)$$

In an interval I the partial

costs are $p^I(\sigma) = \sum_{i: t_i \in I} p_i$

and the partial strategy costs are

$$\mathrm{cost}^I(\sigma) = \sum_{i: t_i \in I} \mathrm{cost}(\sigma_i)$$

We call an interval cheap if

$$p^I(\sigma) < C_{crit}$$

$$C_{crit} = \frac{C}{1+\beta}$$ (otherwise 'expensive')

c) Crit break-even point

(Buying a BC at the beginning of an expensive interval saves money.)

For any request seq. there is a B-schedule of optimal cost $C_{OPT}(\sigma)$, offline sol. not necessarily unique

Obs: Let σ be a request sequence and $T_{OPT}(\sigma)$ an optimal B-schedule. Then we can assume w.l.o.g. that

a) OPT never buys a BC at a reduced request

b) If I is an exp. interval of length at most T then OPT has at least one reduced request in I

(in the offline scenario we can compute an opt. B-schedule for n requests in time $O(n)$)

<u>Def. online alg.</u>

NEVER: $\frac{1}{\beta}$ - comp.

Theorem: No det. alg. performs better than $(2-\beta)$ - comp.

Prove: Let A be an online alg. for $BCC(1, \beta, T)$. Let $\varepsilon > 0$ be arbitrarily small.

↳ as long as we have no BC, we get requests of cost ε (very dense, all in $[0, T)$)

↳ as soon as we buy, no requests anymore

Let S be the accumulated cost of the requests (without the current me)

$$cost_A(\sigma) = C + S + \beta\varepsilon$$

and
$$cost_{OPT}(\sigma) = \begin{cases} S + \varepsilon & \text{if } S+\varepsilon \le \varepsilon_{crit} \\ C + \beta(S+\varepsilon) & \text{otherwise} \end{cases}$$

Hence

$$\frac{cost_A(\sigma)}{cost_{OPT}(\sigma)} = \begin{cases} \dfrac{C+S+\beta\varepsilon}{S+\varepsilon} & \text{if } S+\varepsilon \le \varepsilon_{crit} \\ \dfrac{C+S+\beta\varepsilon}{C+\beta(S+\varepsilon)} & \text{otherwise} \end{cases}$$

$$\ge 2-\beta-\frac{\varepsilon(1-\beta)^2}{C}$$

As ε is arbitrary small $\Rightarrow \ge 2-\beta$ //

SKM alg. achieves this.

Rand. solution is $\dfrac{2}{1+\beta}$ - comp.

Theorem: SKM is $2-\beta$ comp.

Prove: σ, $t_{OPT}(\sigma)$

Divide time into periods

$[\tau_j, \tau_{j+1})$ $\quad 0 \le j \le q$

(assuming $\tau_0 = 0$ and $\tau_{q+1} = \infty$)

Each period starts with an expensive phase $[\tau_j, \tau_j + T)$, followed by a cheap phase $[\tau_j + T, \tau_{j+1})$.

SUM will buy at most one BC during each phase (following from the obs. and the fact that $(t - \tau, t]$ must be an expensive phase)

Clearly $C^{I}_{SUM} \le C^{I}_{OPT}(\sigma)$ for a cheap phase

So let I be an expensive phase.

We divide I in three subphases I_1, I_2, I_3.

In I_1 and I_3 SUM has a valid

BC, but not in T_2

For $i \in \{1,2,3\}$ let

$$s_i = p^{T_i}(\sigma)$$

Then

$$\text{cost}_{\text{sum}} \leq C + s_2 + \beta(s_1 + s_3)$$

and

$$\text{cost}_{\text{opt}} = C + \beta(s_1 + s_2 + s_3)$$

Hence

$$\frac{\text{cost}_{\text{sum}}}{\text{cost}_{\text{opt}}} \leq \frac{C + s_2 + \beta(s_1 + s_3)}{C + \beta(s_1 + s_2 + s_3)}$$

worst case

$$\leq \frac{C + c_{\text{crit}}}{C + \beta c_{\text{crit}}}$$

$$c_{\text{crit}} = \frac{C}{1-\beta} \qquad = 2 - \beta \quad //$$