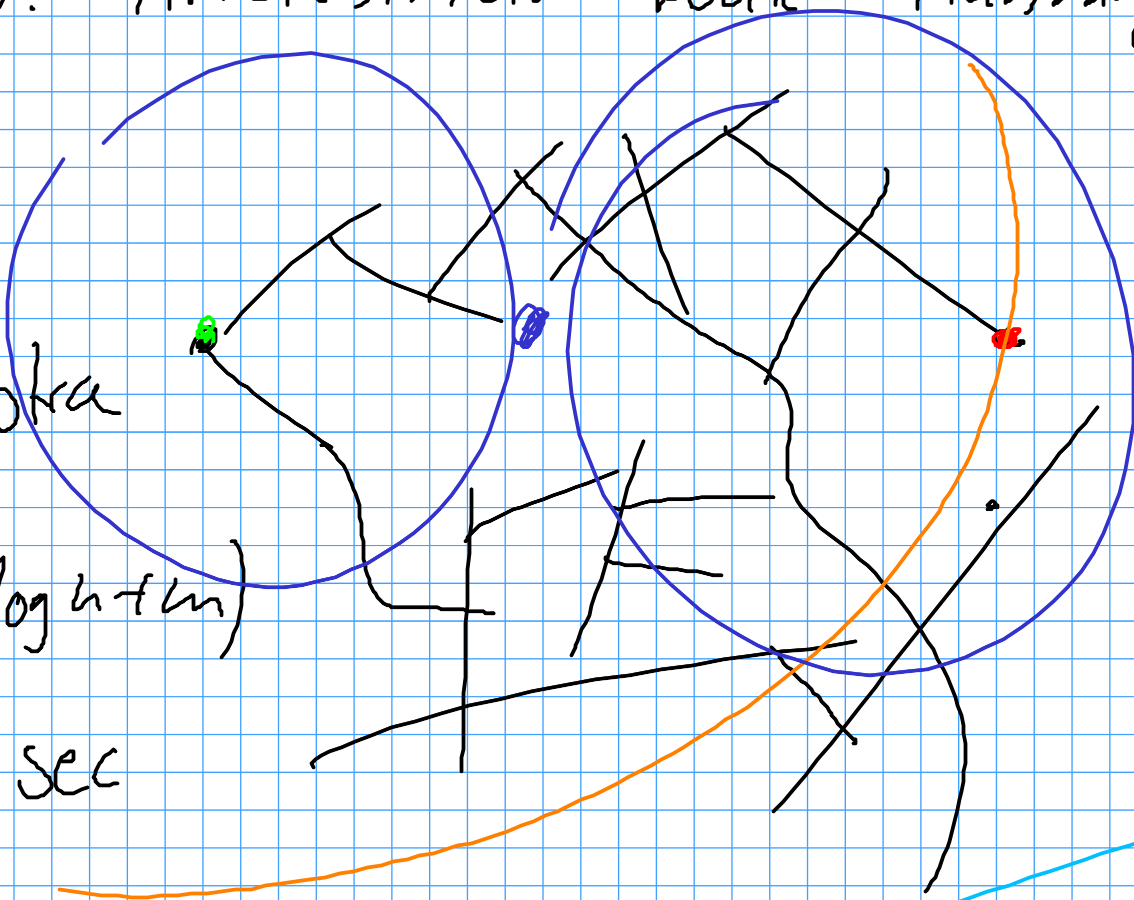
2.6. APPLICATION: Route-Planning

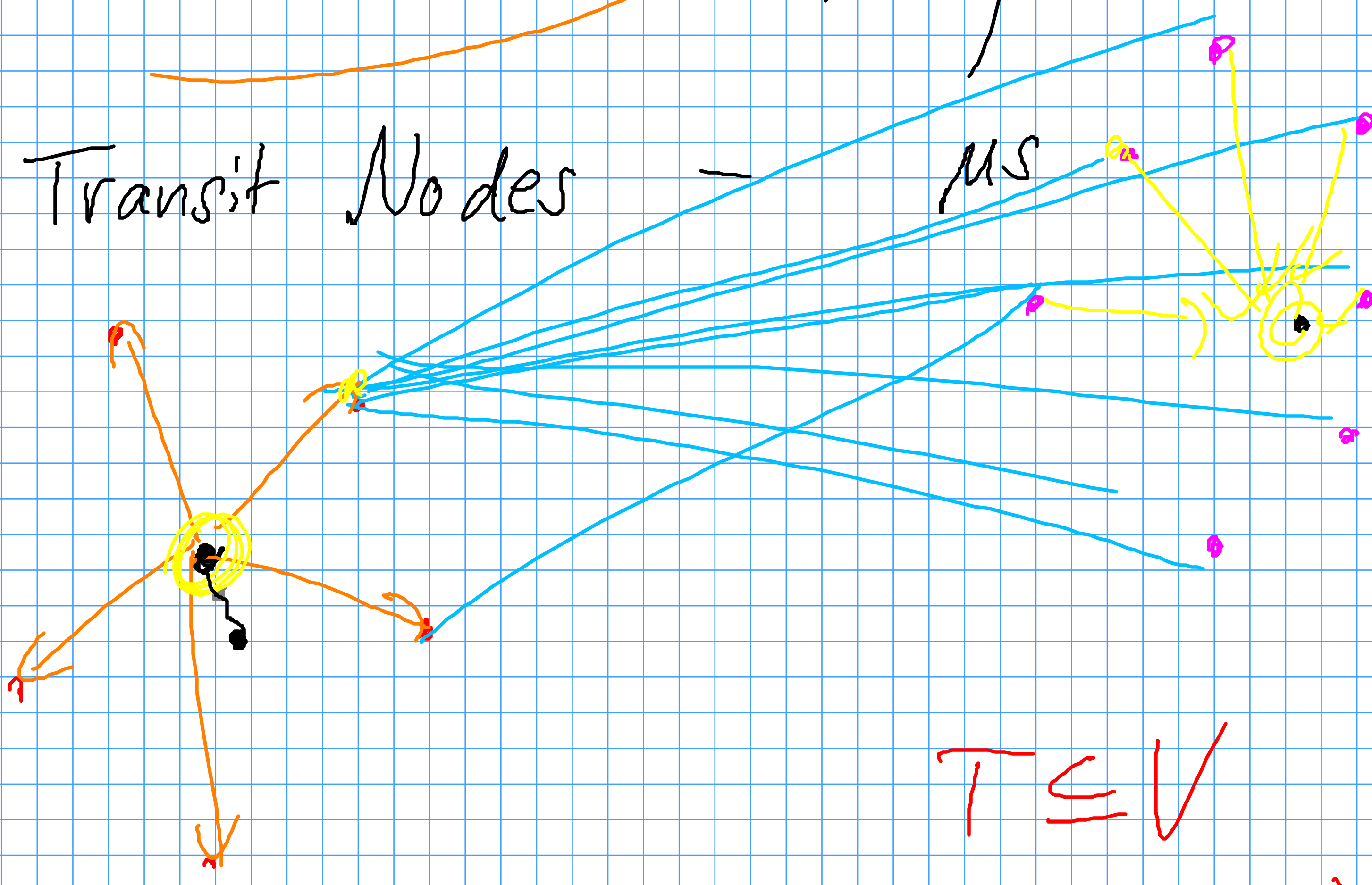


Dijkstra

$O(n \log n + m)$

sec

Transit Nodes — μs

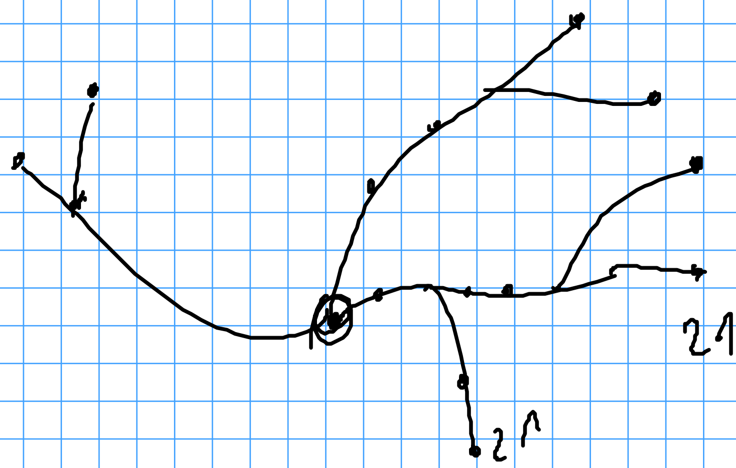


$T \subseteq V$

$|T| \in O(\sqrt{n})$

HittingSet formulation k

- small shortest paths ≤ 20 nodes



Approach:
grow shortest path tree from every $v \in V$ until all leaves are on 'long' paths
→ extract those paths
→ compute global Hitting Set

Want to apply ε-net theorem.

$\varepsilon = \frac{k}{n}$ → Accordingly we can find a transit node set of size $O\left(d \frac{n}{k} \log\left(\frac{n}{k}\right)\right)$ in polytime.

→ undirected unique shortest paths $d=2$

→ directed $d=3$

$O(\log(|OPT|))$ - APX as well

Facility Location Problems

* gas stations: along every shortest path of at least 70km there should be a gas station
* Signs: on interstates, every 20km a sign should indicate cities in driving direction

GOAL: • minimize the number of required objects
• put out locations

Exercise: How to get from distances to hop distances?
→ uniform subsampling
→ compute hop-bound by finding $\max_{e \in E} c(e)$

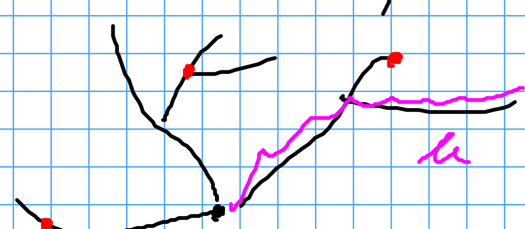
→ Universe: possible driving distances

Two heuristics:

- adaptive sampling
- pruning

Adaptive Sampling

- Hitting Set $C = \emptyset$
- consider $v \in V$ in some (random) order
  - ↳ if v is mandatory to construct a Hitting Set in the end, we add v to C
  - ↳ check: run Dijkstra from v, if there is a shortest path longer than h with no vertex from C on it, v is needed

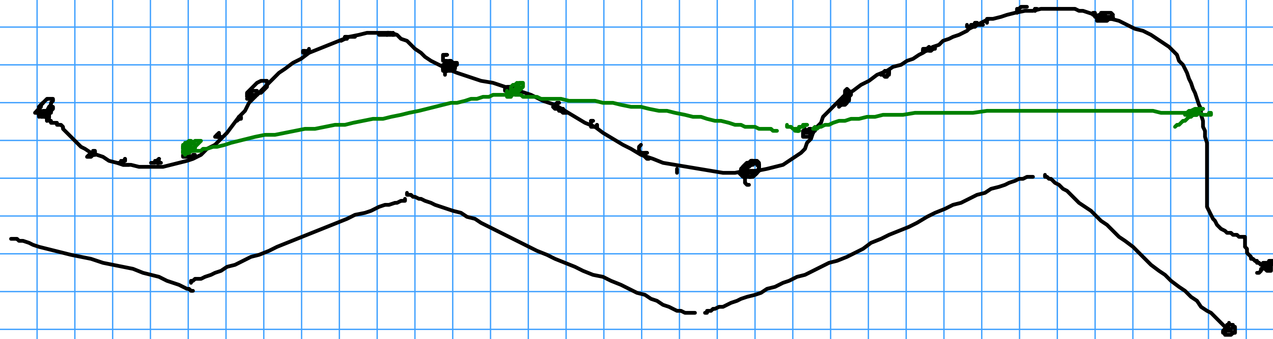


Pruning

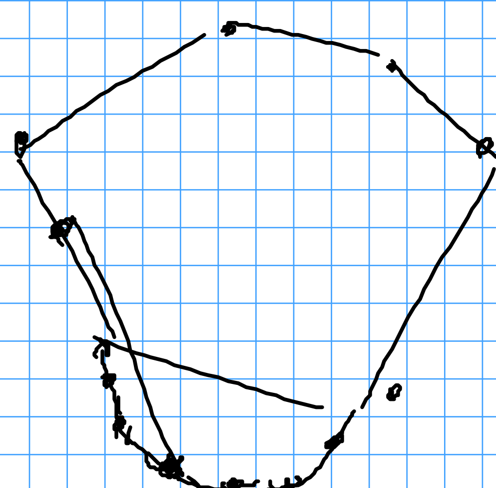
- Hitting Set $C = V$
- consider $v \in V$ in random order
  if v is not needed to let C be a Hitting Set, then remove v from C
  → checker: run Dijkstra, if v is the only node on a shortest path of required length, it has to be maintained

Exercise: Find a small example for Ad. Sampling where the resulting C is not minimal. Prove that pruning always outputs a minimal C.

Graph Simplification

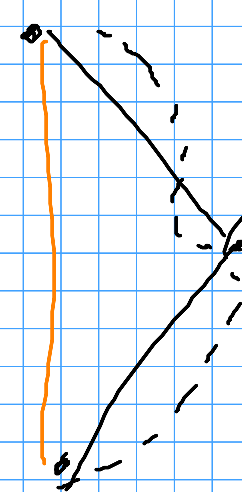


Setting: for a zoom level z we do not want to transmit all vertices in a view, but we think it is sufficient to transmit every $k(z)$-th node on a shortest path



k_1 z_2

$k_1, k_2, \ldots$

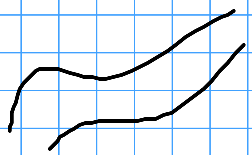


Exercise: To avoid artificial artifacts, we would like to have that the Hitting Set C_1 for ε_1 is a subset of C_2 for ε_2 and so on. Design an algorithm to compute such a sequence of Hitting Sets for given $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$.

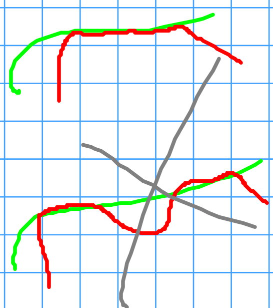
Some Hints for the Golden Exercise

Let $G(V,E)$ be the graph and P_ε the set of all unique shortest paths in G which contain $\varepsilon \cdot n$ nodes (with $n = |V|$). We want to show that $\forall \varepsilon \in [0,1]$ there exists a Hitting Set for P_ε of size $\frac{c}{\varepsilon}$ when c is an ε-independent constant.

Lemma: If $\forall p, p' \in P_\varepsilon$: $p \cap p' = \emptyset$



or $|p \cap p'| \geq \frac{\varepsilon n}{2}$



then we can find a Hitting Set for P_ε of size $\frac{2}{\varepsilon}$.

Proof: P^* – maximal set of paths in P_ε with all paths being intersection free

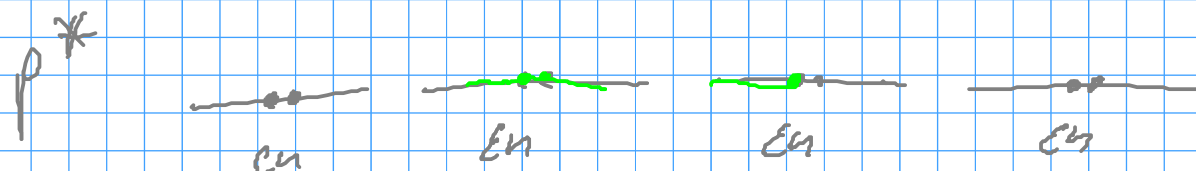
$p \in P_\varepsilon \setminus P^*$ we know that there exists at least one path $p' \in P^*$ with

$$|p \cap p'| \geq \frac{\varepsilon n}{2}$$

$n = 1000$
$\varepsilon = 0.25$
$\varepsilon \cdot n = 250$

How many paths are there in P^*?

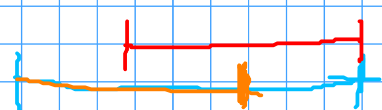
$$\frac{n}{\varepsilon n} = \frac{1}{\varepsilon}$$



Choosing the two points in the middle for every path in P^* gives a HS of size $\leq \frac{2}{\varepsilon}$ and all paths in P_ε are hit automatically. //

Observation: For $\varepsilon > \frac{2}{3}$ this gives us a Hitting Set of size ≤ 2 for P_ε.

↪ Lemma applies automatically.



Lemma: For $\varepsilon \in]\frac{1}{2}, \frac{2}{3}]$, the system P_ε can be hit with 3 nodes

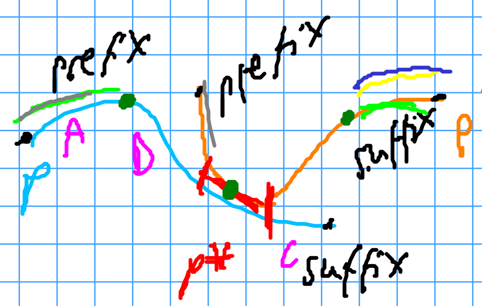
Proof: If the above lemma applies we get a HS of size $\frac{2}{\varepsilon} < \frac{2}{\frac{1}{2}} = 4$

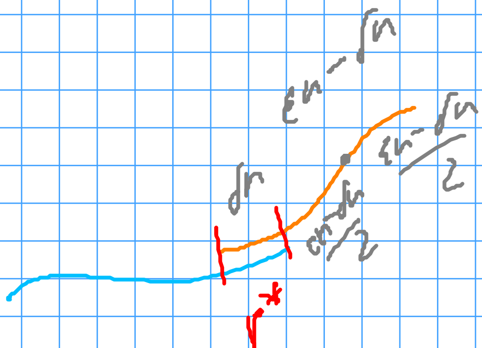
↪ HS size is 3.

Otherwise:

There have to exist two paths $p, p' \in P_\Sigma$:

$$0 < |p \cap p'| < \frac{\epsilon n}{2}$$

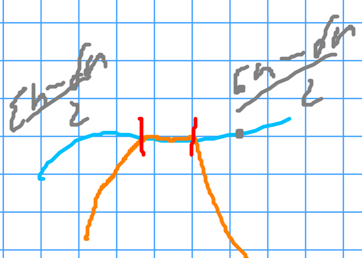




$|p|p^*| = \epsilon n - \delta n$

$|A|, |B|, |C| < \frac{\epsilon n - \delta n}{2}$

$|p^*| = \delta n$



How could an unhit path be composed?

If it intersects p^*:

$$\underbrace{\frac{\delta n}{2}}_{\text{of } p^*} + \underbrace{\frac{\epsilon n - \delta n}{2}}_{p \text{ or } p'} + n - 2\epsilon n + \delta n$$

$$= \frac{\epsilon n}{2} + n - 2\epsilon n + \delta n$$

$$\delta < \frac{\epsilon}{2}$$

$$\le \frac{\epsilon n}{2} + n - 2\epsilon n + \frac{\epsilon}{2} n$$

$= n - \varepsilon n$

$\varepsilon > \frac{1}{2}$

$$\frac{n - \varepsilon n < \varepsilon n}{n < 2\varepsilon n \quad | : 2n}$$

$\underline{\varepsilon > \frac{1}{2}}$

If the white path does not intersect p^* it can have at most:

$2 \cdot \frac{\varepsilon n - \delta n}{2} + n - 2\varepsilon n + \delta n$

p or p'

$= n - \varepsilon n < \varepsilon n$

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