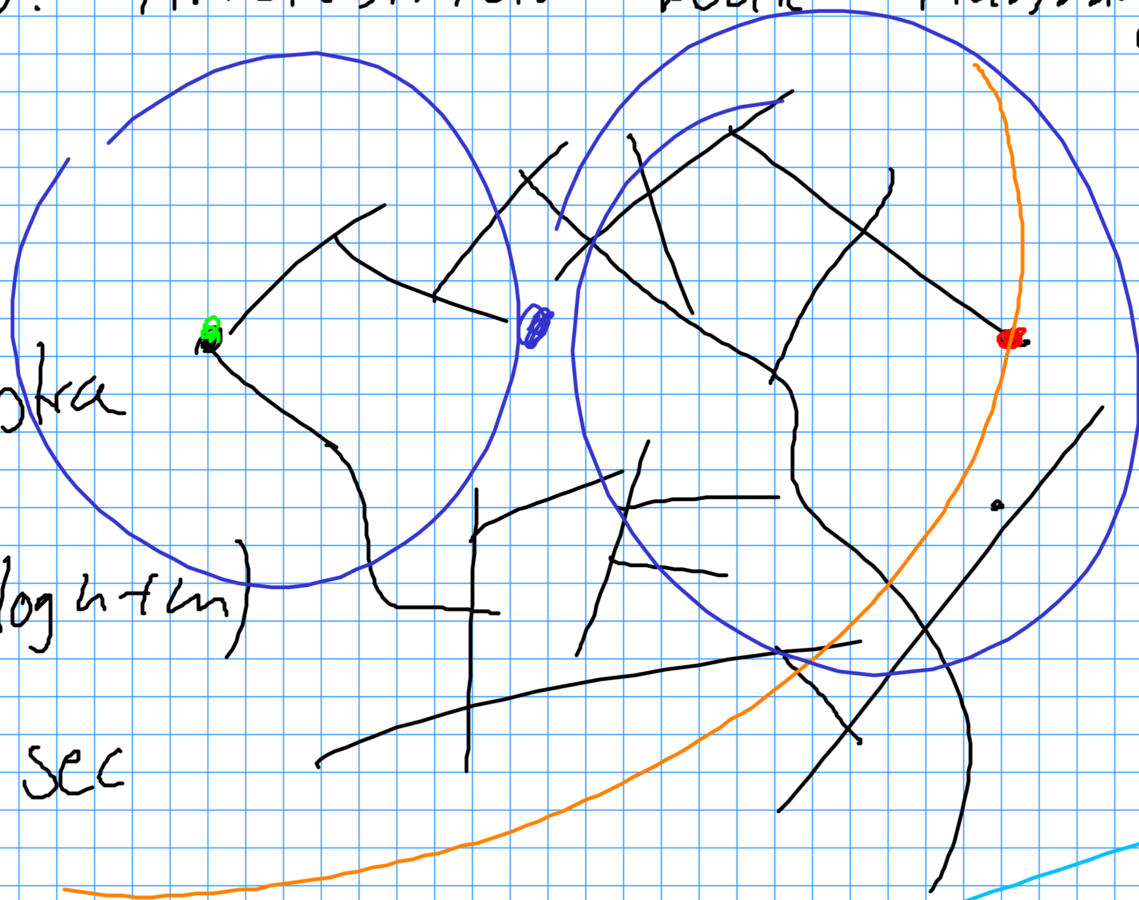
# 2.6. APPLICATION: Route-Planning

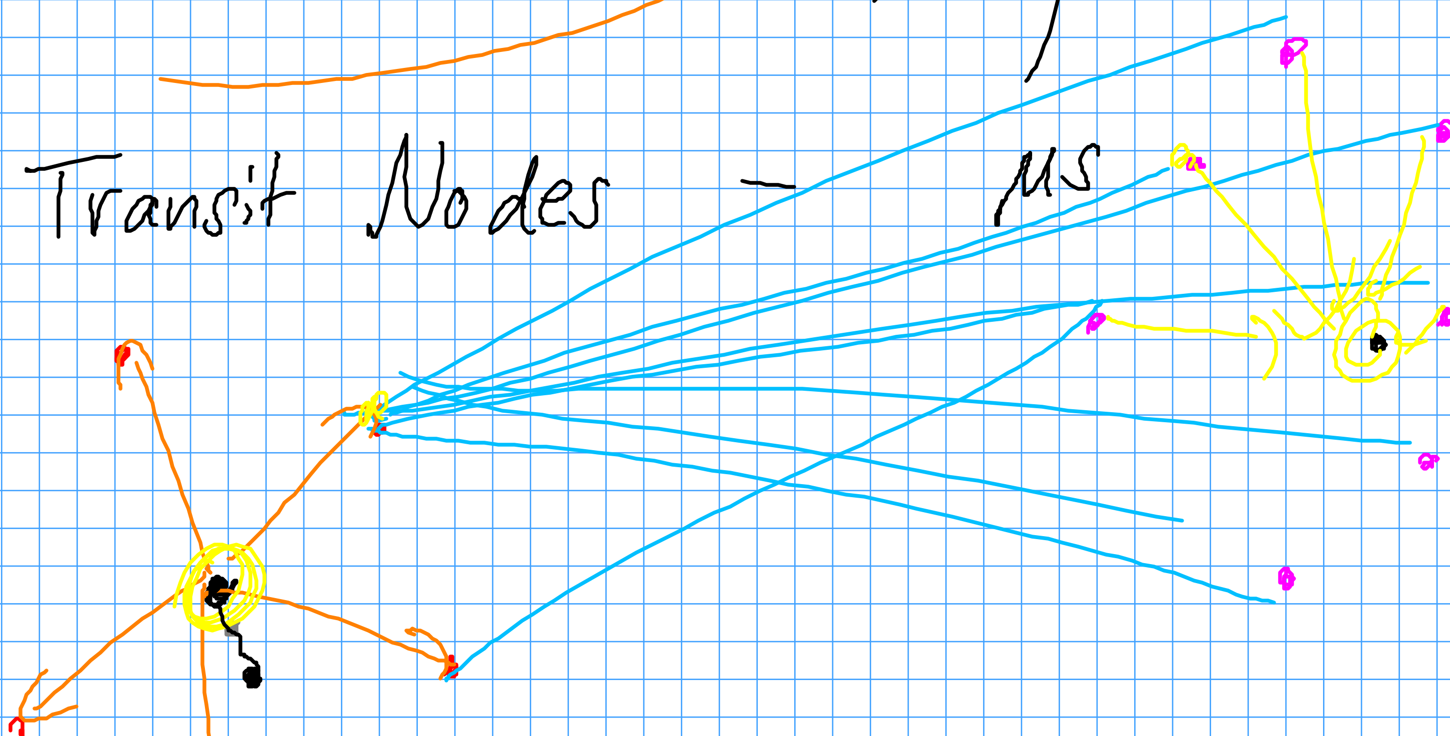


Dijkstra

$O(n \log n + m)$

sec

Transit Nodes — $\mu s$

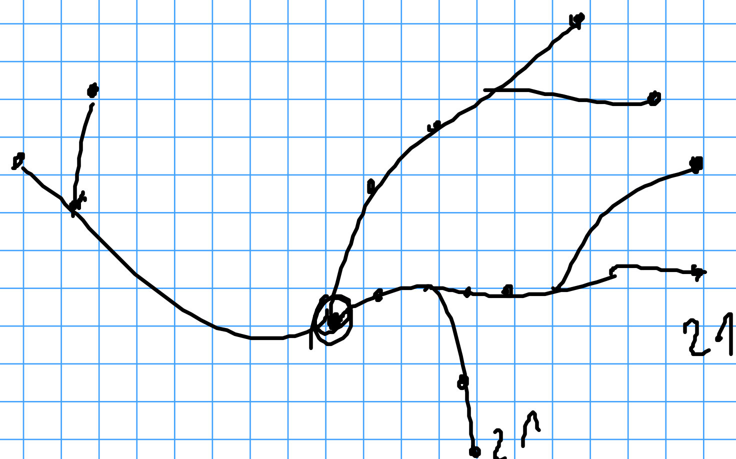


$T \subseteq V$

$|T| \in O(\sqrt{n})$

HittingSet formulation $h$

- small shortest path $\leq 20$ nodes




Approach: grow shortest path tree from every $v \in V$ until all leaves are on 'long' paths
→ extract those paths
→ compute global Hitting Set

Want to apply $\varepsilon$-net theorem.

$\varepsilon = \frac{h}{n}$ → Accordingly we can find a transit node set of size $\mathcal{O}\left(\cancel{d} \frac{n}{h} \log\left(d \frac{n}{h}\right)\right)$ in polytime.

→ undirected unique shortest paths $d=2$

→ directed $d=3$

$O(\log(|OPT|))$ - APX as well

---

Facility Location Problems

* gas stations: along every shortest path of at least 50 km there should be a gas station
* Signs: on interstates, every 20 km a sign should indicate cities in driving direction

GOAL: • minimize the number of required objects
• put out locations

Exercise: How to get from distances to hop distances?
→ uniform subsampling
→ compute hop-bound by finding $\max_{e \in E} c(e)$

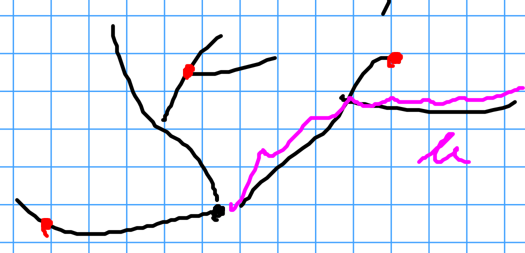
$\longrightarrow$ Universe: possible driving distances

## Two heuristics:

- adaptive sampling
- pruning

## Adaptive Sampling

- HittingSet $C = \emptyset$
- consider $v \in V$ in some (random) order
  - ↳ if $v$ is mandatory to constitute a Hitting Set in the end, we add $v$ to $C$
  - ↳ check: run Dijkstra from $v$, if there is a shortest path longer than $k$ with no vertex from $C$ on it, $v$ is needed

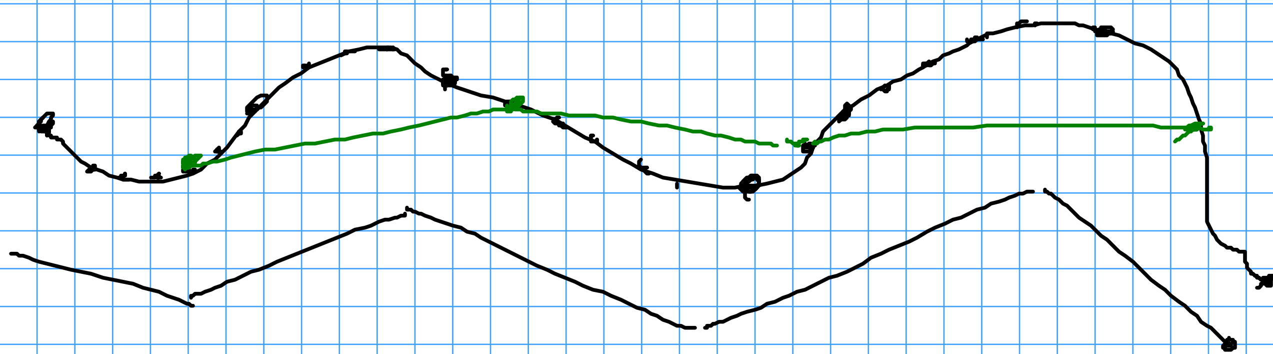


# Pruning

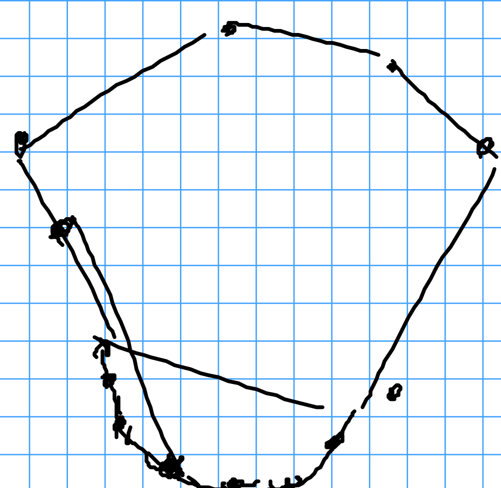
- Hitting Set $C = V$
- consider $v \in V$ in random order
  if $v$ is not needed to
  let $C$ be a Hitting Set, then
  remove $v$ from $C$
  → checker: run Dijkstra, if $v$ is the
  only node on a shortest path
  of required length, it has to
  be maintained

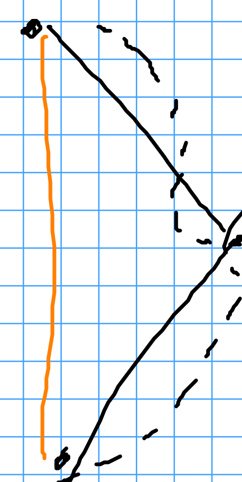
**Exercise:** Find a small example
for Ad. Sampling where the
resulting $C$ is not minimal.
Prove that pruning always
outputs a minimal $C$.

# Graph Simplification



Setting: for a zoom level $z$ we do not want to transmit all vertices in a view, but we think it is sufficient to transmit every $k(z)$-th node on a shortest path





$z_1$ $z_2$

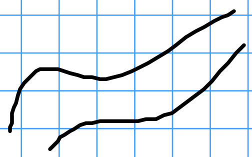
$k_1, k_2, \ldots$

Exercise: To avoid artificial artefacts, we would like to have that the Hitting Set $C_1$ for $\varepsilon_1$ is a subset of $C_2$ for $\varepsilon_2$ and so on. Design an algorithm to compute such a sequence of Hitting Sets for given $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$.

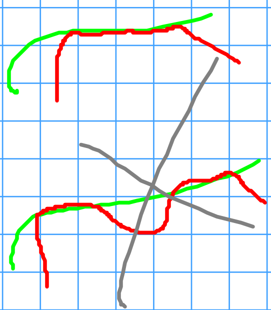
Some Hints for the Golden Exercise

Let $G(V,E)$ be the graph and $P_\varepsilon$ the set of all unique shortest paths in $G$ which contain $\varepsilon \cdot n$ nodes (with $n = |V|$). We want to show that $\forall \varepsilon \in [0,1]$ there exists a Hitting Set for $P_\varepsilon$ of size $\frac{c}{\varepsilon}$ when $c$ is an $\varepsilon$-independent constant.

Lemma: If $\forall p, p' \in P_\varepsilon$ : $p \cap p' = \emptyset$



or $|p \cap p'| \geq \frac{\varepsilon n}{2}$



then we can find a Hitting Set for $P_\varepsilon$ of size $\frac{2}{\varepsilon}$.

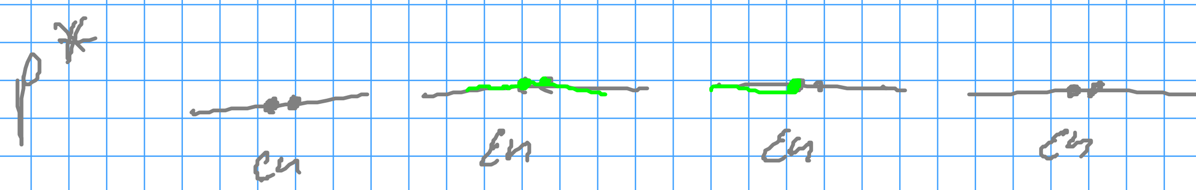
Proof: $P^*$ – maximal set of paths in $P_\varepsilon$ with all paths being intersection free

$p \in P_\varepsilon \setminus P^*$ we know that there exists at least one path $p' \in P^*$ with $|p \cap p'| \geq \frac{\varepsilon n}{2}$

$n = 1000$
$\varepsilon = 0.25$
$\varepsilon \cdot n = 250$

How many paths are there in $P^*$?

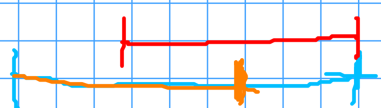
$$\frac{n}{\varepsilon n} = \frac{1}{\varepsilon}$$



Choosing the two points in the middle for every path in $P^*$ gives a HS of size $\le \frac{2}{\varepsilon}$ and all paths in $P_\varepsilon$ are hit automatically.

Observation: For $\varepsilon > \frac{2}{3}$ this gives us a Hitting Set of size $\le 2$ for $P_\varepsilon$.

↳ Lemma applies automatically.



Lemma: For $\varepsilon \in ]\frac{1}{2}, \frac{2}{3}]$, the system $P_\varepsilon$ can be hit with 3 nodes

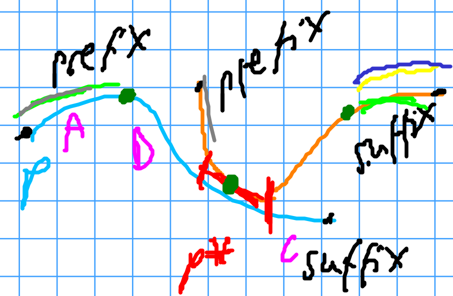
Proof: If the above Lemma applies we get a HS of size $\frac{2}{\varepsilon} < \frac{2}{\frac{1}{2}} = 4$

↳ HS size is 3.

Otherwise:

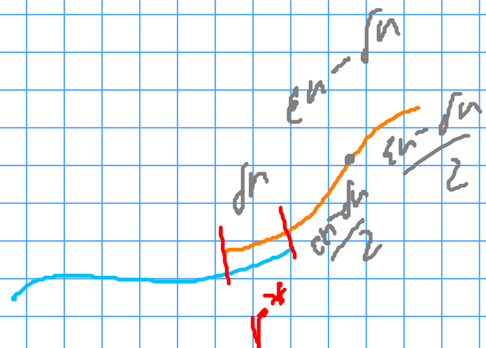
There have to exist two paths $p, p' \in P_\Sigma$:

$$0 < |p \cap p'| < \frac{\varepsilon n}{2}$$

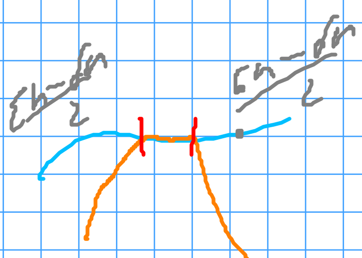



$|p|p^*| = \varepsilon n - \delta n$

$|A|, |B|, |C| < \frac{\varepsilon n - \delta n}{2}$




$|p^*| = \delta n$




How could an init path be composed?

If it intersects $p^*$:

$$\underbrace{\frac{\delta n}{2}}_{\text{of } p^*} + \underbrace{\frac{\varepsilon n - \delta n}{2}}_{p \text{ or } p'} + n - 2\varepsilon n + \delta n$$

$$= \frac{\varepsilon n}{2} + n - 2\varepsilon n + \delta n$$

$$\delta < \frac{\varepsilon}{2}$$

$$\le \frac{\varepsilon n}{2} + n - 2\varepsilon n + \frac{\varepsilon}{2} n$$

$$= n - \varepsilon n$$

$$\varepsilon > \frac{1}{2}$$

$$\frac{n - \varepsilon n < \varepsilon n}{n < 2\varepsilon n} \quad \Big| : 2n$$

$$\underline{\varepsilon > \frac{1}{2}}$$

If the white path does not intersect $p^*$ it can have at most:

$$2 \cdot \frac{\varepsilon n - \delta n}{2} + n - 2\varepsilon n + \delta n$$

$p$ or $p'$

$$= n - \varepsilon n < \varepsilon n$$